Matt Allen
Daniel Rixen
Maarten van der Seijs
Paolo Tiso
Thomas Abrahamsson
Randy Mayes

Substructuring in Engineering Dynamics

– Emerging Numerical and Experimental Techniques –

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Preface

One fundamental paradigm in engineering is to break a structure into simpler components in order to simplify test and analysis. In the numerical world, this concept is the basis for Finite Element discretization and is also used in model reduction through substructuring. In experimental dynamics, substructuring approaches such as Transfer Path Analysis (TPA) are commonly used, although the subtleties involved are perhaps not always adequately appreciated. Recently, there has been renewed interest in using measurements alone to create dynamic models for certain components and then assembling them with numerical models to predict the behavior of an assembly. Substructured models are also highly versatile; when only one component is modified, it can be readily assembled with the unchanged parts to predict the global dynamical behavior. Substructuring concepts are critical to engineering practice in many disciplines, and they hold the potential to solve pressing problems in testing and modeling structures where nonlinearities cannot be neglected.

This book originates from lecture notes created for a short course in Udine, Italy in July 2018. We will review a general framework which can be used to describe a multitude of methods and the fundamental concepts underlying substructuring. The course was aimed at explaining the main concepts as well as specific techniques needed to successfully apply substructuring both numerically (i.e. using finite element models) and experimentally. The course centered around the following topics, which range from classical substructuring methods to topics of current research such as substructuring for nonlinear systems.

1. Introduction and motivation
2. Primal and dual assembly of structures
3. Model reduction and substructuring for linear systems including Guyan and Hurty/Craig-Bamtpon reduction, McNeal, Rubin, Craig-Chang, interface reduction methods and model reduction in the state space.
4. Experimental-Analytical substructuring including frequency based substructuring or impedance coupling, substructure decoupling methods including the transmission simulator method, measurement methods for substructuring, the virtual point transformation, state space substructuring and an overview of finite element model updating (a common alternative to experimental substructuring).
5. Industrial applications and related concepts including Transfer Path Analysis and finite element model updating.
6. Model reduction and substructuring methods for non-linear systems are highlighted with a focus on geometrically non-linear structures and nonlinearities due to bolted interfaces.

This text was designed to provide practicing engineers or researchers such as PhD students with a firm grasp of the fundamentals as well as a thorough review of current research in emerging areas. The reader is expected to have a solid foundation in structural dynamics and some exposure to finite element analysis. The material will be of interest to those who primarily perform finite element simulations of dynamic structures, to those who primarily focus on modal test, and to those who work at the interface between test and analysis.

Udine,
July 2018

Matt Allen
Daniel Rixen
Maarten van der Seijs
Paolo Tiso
Thomas Abrahamsson
Randall Mayes

Contents

Chapter 1
Introduction and Motivation

Abstract "Divide and Conquer" is a paradigm that helped Julius Cesar to dominate on the wide roman Empire. The power of dividing systems, then analyze them as parts before combining them in an assembly, is also an approach often followed in science and engineering. In this introductory chapter, we shortly discuss the main idea behind domain decomposition and substructuring applied to mechanical systems. — *Chapter Author: Daniel Rixen*

1.1 Divide and conquer approaches in the history of engineering mechanics

The work of engineers consists in understanding systems in order to build new solutions or optimize existing ones. To analyze systems, it is very common and efficient to decompose them into subcomponents. This reduces the complexity of the overall problem by considering first its parts and can provide very useful insight for optimizing or troubleshooting intricate structures. In this introductory chapter, the general idea of substructuring is discussed and an overview of the topics treated in these lecture notes is given.

One often refers to Schwarz [254] who proved the existence and uniqueness of the solution of a Poisson problem on a domain that was a combination of a square and a circle. By considering the problem as two separate problems on simple overlapping square and circular domains for which existence and uniqueness of the solution was already known, and devising an iteration strategy between the two domains with guaranteed convergence, he could conclude his very important proof. The algorithm that he devised as a means for his proof is still used today, underlying some very popular solution strategies on multiprocessor computers (see for instance [292]).

But in essence, approximation techniques such as the Rayleigh-Ritz method [82, 225, 229], where the solution space is approximated by a reduced number of admissible functions, can also be considered as divide an conquer methods: the solution space is decomposed into functional subspaces in which an approximate solution can be found. The Finite Element technique, pioneered by mathematicians like Courant [44, 45], can in fact be seen as a clever application of the Rayleigh-Ritz technique on small region of the computational domain, resulting therefore in not only a function decomposition of the problem within the element, but also a spatial decomposition of the problem as the physical domain is broken into elements.

Today when we consider substructuring, most often we envison dividing a finite element model into subcomponents that each contain many elements. This substructuring approach was developed in the sixties, motivated by the need to solve structural problems in aerospace and aeronautics of several tens of nodes (considered to be very large problems at that time, given the fact that computers were just arising). Hence, the idea was to add another level of decomposition and reduction by subdividing the mesh of a finite element model into substructures and to represent the dynamics per substructure in a reduced and approximate way. Some of the substructuring techniques still commonly used today originate from that time where memory and CPU power were very limited.

Later, in the eighties and especially in the nineties, solving very large problems using domain decomposition became a very active research topic. In order to efficiently use the computational power of new multi-processor machines, it was essential to minimize the communications between processing units and to ensure that each CPU was fed with enough work that could be done independently. Domain Decomposition is the paradigm to achieve this, by letting each CPU solve the local problems of each domain

and searching for the global solution by iterating on the interface solutions [292]). Nowadays, high performance computing techniques solve engineering problems of close to a billion degrees of freedom on several hundreds of thousands of processors thanks to that paradigm [290].

The last wave of intensive research in substructuring came at the beginning of the 21st century, this time not directly powered by exponential grows of computing power, but by significant progress in techniques to accurately measure the dynamics of components. Experimental substructuring aims to build models of assemblies from the data measured on components, possibly in combination with numerical models. Although the theory of experimental dynamic substructuring is essentially the same as its analytical counterpart, many novel techniques needed to be developed in order to alleviate the adverse effects of measurement errors (even if strongly reduced thanks to novel sensing and acquisition techniques) when building a model based on experimentally characterized components.

1.2 Advantages of substructuring in mechanical engineering

In mechanical engineering practice, substructuring techniques are useful for the following cases:

- When working on a large project (for example the design of an aircraft), different groups and departments work on different components. In that case, it is essential that each team can concentrate on the modeling of their part, and then have an efficient means for assembling to other components in order to predict the overall dynamics and optimize their design accordingly.
- In industry, it is common that models of subsystems must be shared. A classical example is the coupled load analysis of the assembly of a satellite and a launcher: in that case, the satellite company needs to share its model with the launcher operator in order to perform a global dynamic analysis and guarantee that vibration limits will not be crossed. In that case, exchanging information can be efficient if the different components have been modeled separately in such a way that their complexity (number of degrees of freedom) is low and such that they can be assembled on their interface.
- In many engineering problems, parts of the system to be analyzed are significantly more complex than the rest. In structural dynamics for instance, parts of the model could be highly non-linear whereas the remainder can be considered as linear. In that case, it is very advantageous to substructure the problem in such a way that only the parts that are non-linear are treated with the appropriate techniques, whereas the linear parts are for instance condensed once and for all. Such a subdivision of the problem into substructures of different complexity can greatly improve the analysis speed and give better insight on the dynamic behavior.
- When optimizing the design during the development of a new product, or during an improvement of an old design, one often wants to modify only a small number of parts (for instance change the bushings connecting the power-train of a car to the chassis in order to modify the noise and vibration harshness). When the parts that will not be modified are available as substructures and already precomputed (e.g. their dynamics on the interface is known), then recomputing the global behavior when only the components under optimization are changed requires a significantly reduced cost, enabling much faster design cycles.
- In experimental testing, it is not always possible to test the full system; often system level analysis is needed before all of the components have been manufactured. In other cases it is impossible to test the full assembly because appropriately exciting a large structure would require forces that are beyond the capabilities of existing shakers. It is then very advantageous to experimentally characterize the structure part by part. This allows troubleshooting problems arising from the local dynamics in those parts, but also to build a full model by assembling the measured parts using experimental substructuring techniques.
- When parts of the system have not yet been built, it is possible to combine the measured dynamics of hardware components with other parts that are modeled only numerically (hybrid substructuring). In some cases, the combination in real-time of the transient response of a hardware component (the physical substructure) with the dynamics of a numerical substructure is essential to predict the dynamics of the hardware part in realistic conditions. Such hardware-in-the-loop tests (also called Real-Time Substructuring, Hybrid Testing or Cyber-Physics) allow testing very complex systems by having only a part in the lab, the rest of the system being co-simulated in real time.
- Substructuring methods open new opportunities for efficient analyses and design, including the development of digital twins, or a model that can be used to monitor the heath of a system.

These are only a few examples of cases in which substructuring can be advantageous. For these and other reasons, substructuring has attracted a lot of attention over the years and is yet today a very active research field.

Chapter 2
Preliminaries: primal and dual assembly of dynamic models

Abstract There are several ways to formulate the dynamics of a substructure. The different domains in which the dynamics can be described will be reviewed since the manner in which substructures are characterized will later determine the substructuring methodology that can be applied. In addition to how the substructures are formulated, the way in which the coupling/decoupling problem is expressed will allow us in the subsequent chapters to develop different numerical and experimental techniques[1]. Two conditions must be satisfied on the interface between substructures: a condition on the displacement field (compatibility) and on the interface stresses (force equilibrium). Those conditions can be accounted for following several different formulations, all mathematically equivalent, but each leading to different numerical methods, experimental approaches and approximation techniques, as will be explained in the following chapters. In this chapter we outline the basic concepts of the so-called three field formulation, dual and primal assembly. — *Chapter Author: Daniel Rixen*

2.1 The dynamics of a substructure: domains of representation

In these lecture notes, we assume that the problem has already been discretized (using for instance an appropriate Finite Element or Boundary Element formulation). The discretized problem describing the dynamics in a substructure $\Omega^{(s)}$ can be written in the form

$$\mathbf{M}^{(s)}\ddot{\mathbf{u}}^{(s)} + \mathbf{f}_{nl}^{(s)}\left(\dot{\mathbf{u}}^{(s)}, \mathbf{u}^{(s)}\right) = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} \tag{2.1}$$

The superscript (s) indicates that the equation is written for a given substructure s. $\mathbf{M}$ denotes the discretized mass matrix[2], $\mathbf{f}_{nl}$ the discretized non-linear internal force function, which depends in general on the nodal velocities and displacements $\dot{\mathbf{u}}$ and $\mathbf{u}$. Two types of forces are applied on the substructures: the externally applied forces, denoted $\mathbf{f}$, and the forces due to interactions between substructures, denoted $\mathbf{g}$. The latter are in fact internal forces when considering the entire structure, but are considered as applied forces when analyzing the substructure problem.

In case of small displacements, the internal forces $\mathbf{f}_{nl}$ can be linearized and the substructure dynamics are described by

$$\mathbf{M}^{(s)}\ddot{\mathbf{u}}^{(s)} + \mathbf{C}^{(s)}\dot{\mathbf{u}}^{(s)} + \mathbf{K}^{(s)}\mathbf{u}^{(s)} = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} \tag{2.2}$$

where $\mathbf{C}^{(s)} = \partial \mathbf{f}_{nl}^{(s)} / \partial \dot{\mathbf{u}}^{(s)}$ and $\mathbf{K}^{(s)} = \partial \mathbf{f}_{nl}^{(s)} / \partial \mathbf{u}^{(s)}$ are the linearized (tangential) damping and stiffness matrices typically computed around equilibrium positions of the system [82].

The unknowns $\mathbf{u}$ in equations (2.1,2.2) express the behavior of a substructure in terms of its physical displacements (discretized at nodes) in the time domain. Therefore, we will refer to them as the equations in the *physical and time domain*. The first qualification refers to the spatial meaning of the unknowns whereas

[1] A similar classification can be found in a different form in [146].

[2] Here we assume that the mass matrix is constant. This is not the case for models in multibody dynamics for instance [80], but this will not be considered here.

the second refers to the spectral description of the problem. Other spatial and spectral description of the dynamic problem will now be introduced since the way in which the problem is described has an impact on what method can be applied for reduction or experimental assembly, as will be shown throughout these lecture notes. A summary of the different domains in which the substructure dynamics can be considered is given in Figure 2.1. The substructure data in the different domains can be obtained either from numerical models, from experimentally measured data or from a mix of both. The different aspects are explained next.

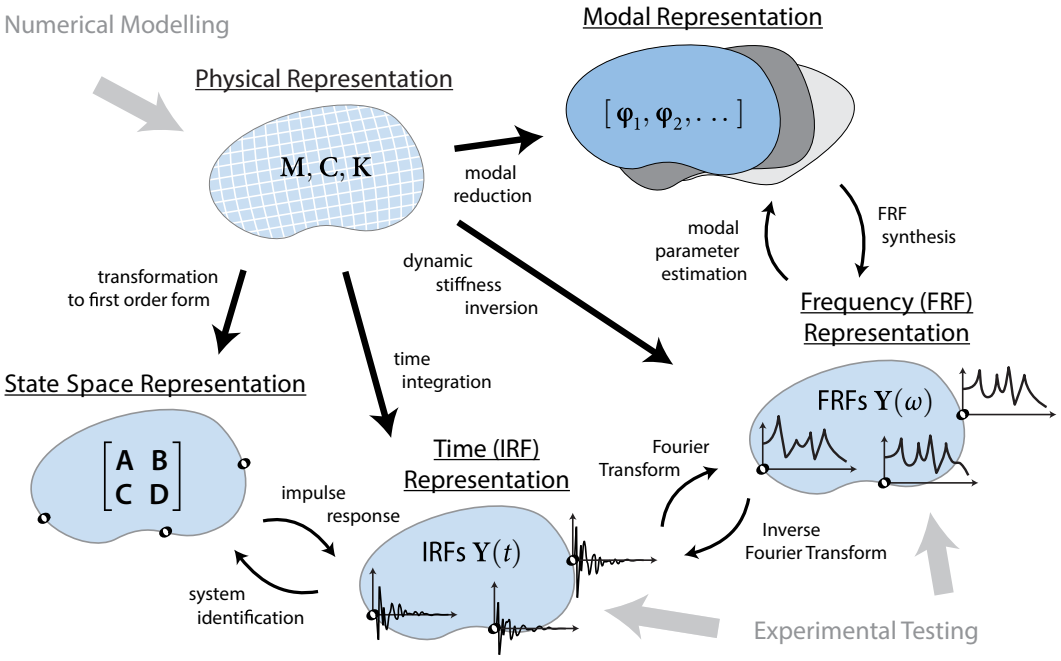

Fig. 2.1: Schematic overview of substructuring domains [259]

2.1.1 Spatial descriptions

Physical domain (continuous and discrete)

The description outlined in (2.2) expresses the dynamics of the substructure using the displacement at specific nodal location and is referred to as a *physical representation*. Note that representation in the physical domain can be either discretized (as assumed in this text) or continuous. In the later case, the unknown is the physical response field and the associated equations are partial differential equations describing the (non-linear) continuous dynamics in a body. This will not be considered here.

Modal domain (reduced and unreduced)

It is often handy and useful to consider the unknowns of a dynamic problem as a combination of vectors of a (sub)space. The most well-known subspace representation is probably the one obtained by mode superposition. The free vibration modes of a substructure defined by the eigenvalue problem [82]

$$\left(\mathbf{K}^{(s)} - \omega_i^{(s)^2}\mathbf{M}^{(s)}\right)\boldsymbol{\phi}_i^{(s)} = \mathbf{0} \tag{2.3}$$

where $\omega_i^{(s)}$ are the eigenfrequencies of the free-free substructure and $\boldsymbol{\phi}_i^{(s)}$ the associated eigenmodes that have the fundamental property of being mass and stiffness orthogonal, namely

$$\boldsymbol{\phi}_i^{(s)^T} \mathbf{K}^{(s)} \boldsymbol{\phi}_j^{(s)} = \omega_i^{(s)^2} \delta_{ij} \tag{2.4}$$

$$\boldsymbol{\phi}_i^{(s)^T} \mathbf{M}^{(s)} \boldsymbol{\phi}_j^{(s)} = \delta_{ij} \tag{2.5}$$

where we have assumed that the modes amplitudes have been chosen to be mass-normalized and where δ_{ij} is the Kronecker symbol such that

$$\delta_{ij} = 1 \quad \text{if} \quad i = j$$
$$\delta_{ij} = 0 \quad \text{if} \quad i \neq j$$

A modal representation of the substructure is then obtained by the change of variable

$$\mathbf{u}^{(s)} = \sum_{i=1}^{n^{(s)}} \boldsymbol{\phi}_i^{(s)} \eta_i^{(s)} = \boldsymbol{\Phi}^{(s)} \boldsymbol{\eta}^{(s)} \tag{2.6}$$

where $n^{(s)}$ is the number of degrees of freedom in substructure (s). Here, $\eta_i^{(s)}$ are the amplitudes of the modal component of the response and are often called modal coordinate of substructure (s). Often we will use a matrix notation as in the second equality of (2.6), where $\boldsymbol{\Phi}^{(s)}$ is a matrix containing in its columns the vibration modes and $\boldsymbol{\eta}^{(s)}$ is a uni-column matrix containing all modal coordinates. Usually only a subset of modes is considered in order to have an approximated but reduced representation of the substructure. This will be discussed in later chapters.

In general, the response $\mathbf{u}^{(s)}$ can be represented as a combination of $n^{(s)}$ independent vectors and we write

$$\mathbf{u}^{(s)} = \sum_{i=1}^{n^{(s)}} \mathbf{v}_i^{(s)} q_i^{(s)} = \mathbf{V}^{(s)} \mathbf{q}^{(s)} \tag{2.7}$$

where $\mathbf{V}^{(s)}$ is a square matrix containing the basis vectors for the desired change of variables. Substituting in the dynamic equation (2.2) and premultiplying by $\mathbf{V}^{(s)^T}$ to project the equations onto the same space, we obtain

$$\mathbf{V}^{(s)^T} \mathbf{M}^{(s)} \mathbf{V}^{(s)} \ddot{\mathbf{q}}^{(s)} + \mathbf{V}^{(s)^T} \mathbf{C}^{(s)} \mathbf{V}^{(s)} \dot{\mathbf{q}}^{(s)} + \mathbf{V}^{(s)^T} \mathbf{K}^{(s)} \mathbf{V}^{(s)} \mathbf{q}^{(s)} = \mathbf{V}^{(s)^T} \mathbf{f}^{(s)} + \mathbf{V}^{(s)^T} \mathbf{g}^{(s)} \tag{2.8}$$

which is usually written as

$$\tilde{\mathbf{M}}^{(s)} \ddot{\mathbf{q}}^{(s)} + \tilde{\mathbf{C}}^{(s)} \dot{\mathbf{q}}^{(s)} + \tilde{\mathbf{K}}^{(s)} \mathbf{q}^{(s)} = \tilde{\mathbf{f}}^{(s)} + \tilde{\mathbf{g}}^{(s)} \tag{2.9}$$

where the tilde superscript indicates that the matrices and vectors pertain now to a representation in a transformed space. The representation vectors stored in $\mathbf{V}^{(s)}$ can be any set of independent vectors, in particular, they can be chosen as the vibration modes $\boldsymbol{\Phi}^{(s)}$, in which case the transformed mass and stiffness matrices $\tilde{\mathbf{M}}^{(s)}$ and $\tilde{\mathbf{K}}^{(s)}$ will be diagonal.

The representation (2.9) will often be referred to as the *modal representation*, even when the base vectors are not vibration modes $\boldsymbol{\Phi}^{(s)}$ but general representation modes $\mathbf{V}^{(s)}$. The associated degrees of freedom $\dot{\mathbf{q}}^{(s)}$ are then called generalized degrees of freedom or modal coordinates and do in general not represent the solution at a particular physical location.

In case an incomplete basis is used for the representation, namely when fewer modes than the number $n^{(s)}$ of degrees of freedom in the substructure are used, the modal representation represents the dynamics in a reduced subspace and in general only in an approximate way (this will be discussed in detail in Chapter 3). We will then call the representation a *reduced modal representation*.

2.1.2 Spectral representation

Time domain (continuous and discrete)

In the form of Eq. (2.2), the unknown dynamic response is considered a function of time and we say that, from a spectral point of view, the equations are expressed in the time domain.

The dynamic equation (2.2) considers the spatial unknowns to be continuous function of time and the dynamics are expressed as ordinary differential equations in time (the space having already be discretized in that equation). As was done for the spatial domain, time can also be discretized using methods related to finite differences (typically variants of the Newmark time integration scheme in structural dynamics [33, 41, 82, 196], but other approaches can also be applied such as Finite Elements in time or variational based approaches [158]). When discretized in time, we will still consider the resulting equations as being in the time domain since the unknowns are the responses at discrete time instances. The equations are then algebraic equations that are typically solved in a recursive form (time stepping), given the fact that the time problem is typically an initial value problem[3].

Frequency domain (reduced and unreduced)

Similar to the decomposition of the spatial response in component modes, the time dependency of the response can also be decomposed into a combination of time contributions. The most classical one is the Fourier decomposition[4] that writes the time function of the response in terms of harmonic functions. Using complex number notations, the Fourier decomposition can be written as

$$\mathbf{u}^{(s)}(t) = \int_{-\infty}^{\infty} \bar{\mathbf{u}}^{(s)}(\omega)\ \mathrm{e}^{-i\omega t} d\omega \tag{2.10}$$

where i is to be understood as the imaginary unit number. This decomposition is very suitable for linear systems since replacing in (2.2) and using the orthogonality properties of harmonic functions, the harmonic component $\mathbf{u}(\omega)$ can be computed individually from the harmonic dynamic equation:

$$\left(-\omega^2\mathbf{M}^{(s)} + i\omega\mathbf{C}^{(s)} + \mathbf{K}^{(s)}\right)\bar{\mathbf{u}}^{(s)} = \bar{\mathbf{f}}^{(s)} + \bar{\mathbf{g}}^{(s)} \qquad \omega \in]-\infty, +\infty[\tag{2.11}$$

where $\bar{\mathbf{f}}^{(s)}$ and $\bar{\mathbf{g}}^{(s)}$ are the Fourier components of the forces, for instance

$$\bar{\mathbf{f}}^{(s)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{f}^{(s)}(t)\ \mathrm{e}^{i\omega t} dt \tag{2.12}$$

The dynamic equation in the frequency domain (2.13) is also often written as

$$\mathbf{Z}^{(s)}\bar{\mathbf{u}}^{(s)} = \bar{\mathbf{f}}^{(s)} + \bar{\mathbf{g}}^{(s)} \qquad \text{where} \quad \mathbf{Z}^{(s)}(\omega) = -\omega^2\mathbf{M}^{(s)} + i\omega\mathbf{C}^{(s)} + \mathbf{K}^{(s)} \tag{2.13}$$

$\mathbf{Z}^{(s)}$ is a *dynamic stiffness matrix* and is a function of the frequency ω. In this form, $\bar{\mathbf{u}}^{(s)}$ is the complex amplitude of the harmonic displacement response (or equivalently the Fourier component of the transient response). A similar equation can be written for the amplitude of the harmonic velocity or accelerations in which case the operator $\mathbf{Z}(\omega)$ is commonly called *the mechanical impedance or the apparent mass* respectively. The dynamic relation can also be inverted and written

$$\bar{\mathbf{u}}^{(s)} = \mathbf{Y}^{(s)}\left(\bar{\mathbf{f}}^{(s)} + \bar{\mathbf{g}}^{(s)}\right) \qquad \text{where} \quad \mathbf{Y}^{(s)}(\omega) = \mathbf{Z}^{(s)}(\omega)^{-1} = \left(-\omega^2\mathbf{M}^{(s)} + i\omega\mathbf{C}^{(s)} + \mathbf{K}^{(s)}\right)^{-1} \tag{2.14}$$

$\mathbf{Y}^{(s)}$ is a Frequency Response Function (FRF) matrix and is often called the *admittance or dynamic flexibility*, or more specifically *receptance, mobility or accelerance/inertance* if $\bar{\mathbf{u}}^{(s)}$ are displacements, velocities or accelerations respectively.

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[3] For an interesting matrix description of time discretization see [241, 299].

[4] Note that other base functions in time can be used (such as wavelets), but this will not be discussed here.

Obviously, in practice, the harmonic components are calculated only for a finite discrete number of frequencies ω, and (2.10) is approximated by the Discrete Fourier Decomposition

$$\mathbf{u}^{(s)}(t) = \sum_{k=-N_\omega}^{N_\omega} \bar{\mathbf{u}}_k^{(s)} \ \mathrm{e}^{i\omega_k} \tag{2.15}$$

choosing a frequency range covering the spectral range of the excitation. It is noteworthy that the decomposition in (2.15) is comparable to the decomposition of the space function of the response in (2.6) and can also be seen as a reduction of the transient response in the time domain. It is an approximation unless the excitation can be exactly represented by a finite combination of harmonics.

Laplace domain

Another often used representation of the time evolution of the dynamic response is in terms of the Laplace components. The idea is to look for the dynamic repsonse when modulated with a decreasing exponential function, namely

$$\bar{\mathbf{u}}^{(s)}(s) = \mathscr{L}\left(\mathbf{u}^{(s)}(t)\right) = \int_0^\infty \mathrm{e}^{-st}\mathbf{u}^{(s)}(t)\ dt \tag{2.16}$$

This transformation changes the differential equation in time into an algebraic equation in the Laplace variable s thanks to the fact that Laplace transforms of time derivatives of $\bar{\mathbf{u}}^{(s)}(t)$ can be written in terms of $\bar{\mathbf{u}}^{(s)}(s)$ using integration by parts. For instance

$$\mathscr{L}\left(\ddot{\mathbf{u}}^{(s)}(t)\right) = s^2\bar{\mathbf{u}}^{(s)}(s) - s\mathbf{u}^{(s)}(t=0) - \dot{\mathbf{u}}^{(s)}(t=0)$$

Clearly, there is a similarity between Laplace and Fourier transforms since (2.16) becomes a Fourier transform if s is taken as imaginary. The main difference is that the inverse transform is trivial for the Fourier domain (leading to the frequency domain decomposition (2.10) or (2.15)) whereas finding the inverse Laplace transform is far more difficult and not general. In structural dynamics, Laplace transforms are used for highly transient problems that can not efficiently be represented by harmonic superposition, such as impact responses and shock propagations (see for instance section 4.3.2 in [82]).

2.1.3 State representation

Displacement space

In addition to represent to space and spectral behavior of the system in different domains as explains above, the very state of the system can be described in mainly two different manners: either one sees the displacements as the only independent unknowns (velocities and accelerations being dependent on the displacement through derivatives) or the velocities are seen as additional independent variables for which the derivative relation to the displacement is explicitly expressed in the formulation. For the first approach, often used in structural dynamics, the dynamics of the system are described by a single set of second order equations as in (2.1) (for non-linear structures) or (2.2) (for linear ones).

State Space representation

In the second case, namely the velocities are seen as additional independent variables, the state of the system is described both by the displacements and the velocities:

$$\mathbf{x}^{(s)}(t) = \begin{bmatrix} \mathbf{u}^{(s)}(t) \\ \dot{\mathbf{u}}^{(s)}(t) \end{bmatrix} \tag{2.17}$$

and the associated linear dynamic equation can for instance be written as

$$\begin{bmatrix} \mathbf{I}^{(s)} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{(s)} \end{bmatrix} \dot{\mathbf{x}}^{(s)} = \begin{bmatrix} \mathbf{0} & \mathbf{I}^{(s)} \\ -\mathbf{K}^{(s)} & -\mathbf{C}^{(s)} \end{bmatrix} \mathbf{x}^{(s)}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{f}^{(s)} + \mathbf{g}^{(s)} \end{bmatrix} \tag{2.18}$$

In this *state-space representation* the number of equations has doubled, but the order of the differential is now reduced to one. This representation is often used especially in control. This form is also commonly used in structural dynamics when strong damping is present since the concept of modes of vibration properly generalizes only when writing the system in the State Space (see for instance section 3.3 in [82]).

2.1.4 Summary of representation domains

From the short summary of the formulation of the structural dynamics problem, it is clear that many variants to describe the problem, combining a spatial, spectral and state representation, can be constructed. In Fig. 2.1, the different aspects are shown graphically and relations between them are summarized. Depending on the representation chosen, numerical and experimental techniques in substructuring can significantly differ as will be seen in these lecture notes.

2.2 Interface conditions for coupled substructures

Let us consider again the linearized dynamic equilibrium equation (2.11) of a substructure in the physical space and in the frequency domain[5]

$$\mathbf{Z}^{(s)}\bar{\mathbf{u}}^{(s)} = \bar{\mathbf{f}}^{(s)} + \bar{\mathbf{g}}^{(s)} \qquad\qquad s = 1 \dots N^{sub} \tag{2.19}$$

where N^{sub} is the total number of substructures in the system.

It is common to write the equilibrium of all substructures in a block matrix form as

$$\mathbf{Z}\bar{\mathbf{u}} = \bar{\mathbf{f}} + \bar{\mathbf{g}} \tag{2.20}$$

with the definitions

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}^{(1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Z}^{(N^{sub})} \end{bmatrix} \tag{2.21}$$

$$\bar{\mathbf{u}} = \begin{bmatrix} \bar{\mathbf{u}}^{(1)} \\ \vdots \\ \bar{\mathbf{u}}^{(N^{sub})} \end{bmatrix} \qquad \bar{\mathbf{f}} = \begin{bmatrix} \bar{\mathbf{f}}^{(1)} \\ \vdots \\ \bar{\mathbf{f}}^{(N^{sub})} \end{bmatrix} \qquad \bar{\mathbf{g}} = \begin{bmatrix} \bar{\mathbf{g}}^{(1)} \\ \vdots \\ \bar{\mathbf{g}}^{(N^{sub})} \end{bmatrix}$$

The dimension of these block matrices and block vectors is $(\sum_s n^{(s)}) \times (\sum_s n^{(s)})$ and $(\sum_s n^{(s)}) \times 1$ respectively.

Since the substructures are part of a same assembly, two interface conditions need to be satisfied: *interface equilibrium and compatibility*. [6]

[5] Expressing the coupling of substructures in other domains (modal, time, state space ...) will be discussed in later chapters and use exactly the same approach.

[6] These lecture notes deal with structural problems. Nevertheless, the general theory is also applicable to the coupling of other physical domains such as acoustics or thermal problems.

2.2.1 Interface equilibrium

The interface equilibrium requires that the interface forces, $\tilde{\mathbf{g}}^{(s)}$, which are internal forces between the substructures, sum to zero when assembled. This is merely a manifestation of Newton's "actio-reactio" principle. Considering for instance an interface $\Gamma^{(sr)}$ between two substructures s and r, one could express this condition as[7]

$$\mathbf{g}_b^{(s)} + \mathbf{g}_b^{(r)} = 0 \qquad \text{on} \quad \Gamma^{(sr)} \tag{2.22}$$

$$\mathbf{g}_i^{(s)} = 0 \qquad \mathbf{g}_i^{(r)} = 0 \tag{2.23}$$

where the subscript b indicates a restriction of the DOF to the boundary and where we assumed that the DOF are numbered in the same manner on both sides of the interface. The subscript i denotes DOF that are not on a boundary and are thus internal DOF. On internal DOF, no connecting forces should exist.

In practice, the numbering of the DOF on the interface will not match across the interfaces and in addition more than two substructures can intersect on an interface (so-called *cross-points* in 2 and 3-D, and edges in 3-D). Hence in general, the interface equilibrium condition needs to be expressed using Boolean localization matrices $\mathbf{L}^{(s)^T}$ of dimension $n \times n^s$ that combine the forces on either side of the interface to satisfy force equilibrium. Interestingly, these localization matrices also map the DOF of substructure s to a global and unique set of n global DOF, as will be elaborated later. In general, the interface equilibrium thus is written as

$$\boxed{\sum_{s=1}^{N^{sub}} \mathbf{L}^{(s)^T} \tilde{\mathbf{g}}^{(s)} = \mathbf{0}} \tag{2.24}$$

This equilibrium condition can also be written using the block matrix $\mathbf{L}^T$ of dimension $n_b \times \sum_s n^s$ acting on the set of all substructure interface forces

$$\boxed{\mathbf{L}^T \tilde{\mathbf{g}} = \mathbf{0}} \qquad \text{where} \quad \mathbf{L}^T = \left[\mathbf{L}^{(1)^T} \ \cdots \ \mathbf{L}^{(N^{sub})^T}\right] \tag{2.25}$$

► To illustrate the notation, consider the examples in Figure 2.2. The green numbers indicate the global nodes for which the interface force conditions are written (i.e. the row number in the localization matrices). For the beam example in Figure 2.2.a, the localization matrices are

$$\mathbf{L}^{(1)^T} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{L}^{(2)^T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{L}^{(2)^T} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the interface equilibrium condition can be written as

[7] In general, we will assume that the decomposition in substructures generates an interface discretization that is conforming and matching, namely that the shape functions used on either side of the interface are identical and that the nodes coincide. In case of non-conforming or non-matching interfaces, the theory used in these lecture notes are still generally applicable, but the assembly operators are then no longer Boolean. Indeed, going back to the variational principle underlying the discretized problem, the part related to the compatibility condition over an interface Γ can be written as

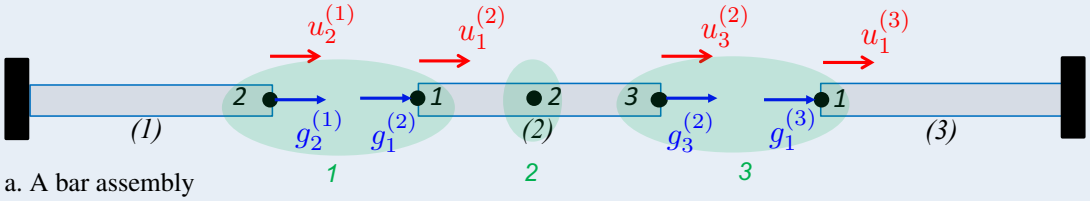
$$\int_\Gamma \mu^T \mathbf{v} d\Gamma$$

where μ and $\mathbf{v}$ are the field. More details about non-matching interfaces can be found, for instance, in [230]

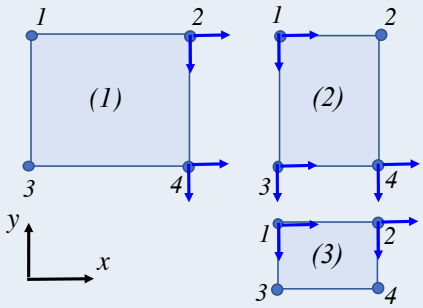
$$\mathbf{L}^T\bar{\mathbf{g}} = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix}\bar{g}_2^{(1)}\end{bmatrix} \\ \begin{bmatrix}\bar{g}_1^{(2)}\\ \bar{g}_2^{(2)}\\ g_3^{(2)}\end{bmatrix} \\ \begin{bmatrix}\bar{g}_1^{(3)}\end{bmatrix} \end{bmatrix} = \mathbf{0} \tag{2.26}$$

For the second example in Figure 2.2.b, we have two degrees of freedom per node and the Localization matrix can be written as

$$\mathbf{L}^T = \begin{bmatrix}\mathbf{L}^{(1)} & \mathbf{L}^{(2)} & \mathbf{L}^{(3)}\end{bmatrix} = \left[\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix} \begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix} \begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\right]$$



a. A bar assembly



b. A 2D assembly

Fig. 2.2: Examples of assemblies: substructure DOFs and interface forces.

These matrices are in fact identical to the localization matrices used in Finite Element codes to assemble elementary matrices in the global system, however here the localization is not written for one element but for one substructure (that one could consider as a super-element or macro-element). Obviously, it is not efficient to store these Boolean matrices as written above, but rather one should store them as sparse matrices, or

even better one should construct the mapping tables based on the connectivity of the substructures over the interfaces.

2.2.2 *Interface compatibility*

The second condition that needs to be satisfied on the interface is that DOF pertaining to the some structural node have the some response on both sides of the interface, or in other words that the DOF are compatible on the interface. Considering the DOF of two substructures s and r coupled on the interface $\Gamma^{(sr)}$, the compatibility condition becomes

$$\bar{\mathbf{u}}_b^{(s)} - \bar{\mathbf{u}}_b^{(r)} = \mathbf{0} \qquad \text{on} \quad \Gamma^{(sr)}$$

where, as before, the subscript b indicates that the compatibility is written for the boundary DOF and where we assumed that the DOF are numbered identically on both sides of the interface.

In general the numbering on the interface does not coincide and therefore the compatibility conditions are expressed using signed Boolean matrices $\mathbf{B}^{(s)}$. When operating on $\bar{\mathbf{u}}^{(s)}$, these operators extract the interfaces DOF and give them an opposite sign on each side of the interface. The interface compatibility can then be written in the following general form:

$$\boxed{\sum_{s=1}^{N^{sub}} \mathbf{B}^{(s)} \bar{\mathbf{u}}^{(s)} = \mathbf{0}} \tag{2.27}$$

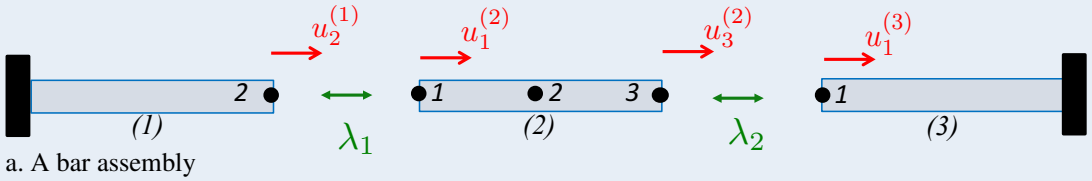
One can use a block matrix notation to write this condition also in the form

$$\boxed{\mathbf{B}\bar{\mathbf{u}} = \mathbf{0}} \qquad \text{where} \quad \mathbf{B} = \left[\mathbf{B}^{(1)} \cdots \mathbf{B}^{(N^{sub})}\right] \tag{2.28}$$

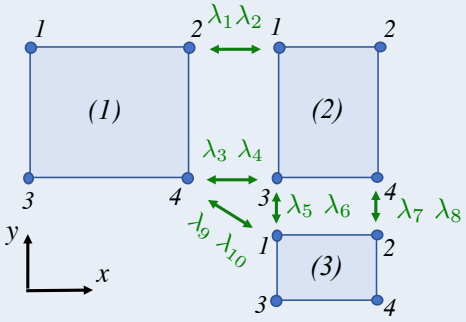
These equations can be understood as compatibility constraints imposed onto the independent sets of DOF in the substructures. The matrices $\mathbf{B}^{(s)}$ have dimension $n_\lambda \times n^{(s)}$, where n_λ is the number of interface compatibility constraints that need to be imposed.

▸ *Example: Boolean Compatibility Matrix*

To illustrate this notation, consider again the examples of Figure 2.2.



a. A bar assembly



b. A 2D assembly

Fig. 2.3: Examples of assemblies: interpretation of the Lagrange multipliers

For the beam example, the compatibility condition can be written as

$$\mathbf{B}\bar{\mathbf{u}} = \begin{bmatrix}\mathbf{B}^{(1)} & \mathbf{B}^{(2)} & \mathbf{B}^{(3)}\end{bmatrix}\bar{\mathbf{u}} = \begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}-1&0&0\\0&0&1\end{bmatrix} & \begin{bmatrix}0\\-1\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\bar{u}_2^{(1)}\end{bmatrix}\\ \begin{bmatrix}\bar{u}_1^{(2)}\\ \bar{u}_2^{(2)}\\ u_3^{(2)}\end{bmatrix}\\ \begin{bmatrix}\bar{u}_1^{(3)}\end{bmatrix}\end{bmatrix} = \mathbf{0} \tag{2.29}$$

The side on which the entry in $\mathbf{B}$ is positive and negative can be chosen freely. The interpretation of $\mathbf{B}$ and its associated Lagrange multipliers is depicted in Figure 2.3.a.

For the second example in Figure 2.2, the constraint matrix can be written as

$$\mathbf{B} = \begin{bmatrix}\mathbf{B}^{(1)} & \mathbf{B}^{(2)} & \mathbf{B}^{(3)}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix} & \begin{bmatrix}-1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&-1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix} & \begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\-1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\-1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\end{bmatrix}\end{bmatrix}$$

The interpretation of $\mathbf{B}$ and its associated Lagrange multipliers is depicted in Figure 2.3.b. Note that the last two constraints are redundant. The compatibility of node 4 in $\Omega^{(1)}$ with node 3 in $\Omega^{(2)}$, and of node 1 in $\Omega^{(3)}$ with node 3 in $\Omega^{(2)}$ is imposed in lines 3, 4, 5 and 6 of $\mathbf{B}$, so there is no need to impose in addition the compatibility between node 4 in $\Omega^{(1)}$ and node 1 in $\Omega^{(3)}$. Adding this redundant constraint does usually not harm the computation and can even be helpful for instance in the parallel computing algorithms .

◀

As for the localization matrix $\mathbf{L}^{(s)}$, the constraint matrices $\mathbf{B}^{(s)}$ are, in practice, not stored as full but as sparse, or only connectivity information is used to apply them to a vector.

To summarize this section, an assembly of substructures in the physical frequency domain (and similarly in other domains) is obtained by imposing compatibility and interface equilibrium conditions, leading to the set of equations

$$\boxed{\begin{cases}\mathbf{Z}^{(s)}\bar{\mathbf{u}}^{(s)} = \bar{\mathbf{f}}^{(s)} + \bar{\mathbf{g}}^{(s)} & s = 1\dots N^{sub}\\ \sum_{s=1}^{N^{sub}} \mathbf{B}^{(s)}\bar{\mathbf{u}}^{(s)} = \mathbf{0}\\ \sum_{s=1}^{N^{sub}} \mathbf{L}^{(s)^T}\bar{\mathbf{g}}^{(s)} = \mathbf{0}\end{cases}} \tag{2.30}$$

or in block matrix form

$$\boxed{\begin{cases}\mathbf{Z}\bar{\mathbf{u}} = \bar{\mathbf{f}} + \bar{\mathbf{g}}\\ \mathbf{B}\bar{\mathbf{u}} = \mathbf{0}\\ \mathbf{L}^T\bar{\mathbf{g}} = \mathbf{0}\end{cases}} \tag{2.31}$$

2.3 Primal and dual assembly

The form (2.30) (or equivalently (2.31)) of the coupled problem uses two interface fields, namely the *primal unknowns* $\mathbf{u}$ per substructure (i.e. on each side of the interfaces) and the substructure interface forces $\mathbf{g}$ called *dual unknowns*.[8] Solving the dynamic problem of the assembly in the form (2.30) can be expensive since many interface unknowns need to be resolved. Hence, these equations can be rearranged in order to eliminate the interface forces and write the problem in terms of unique interface displacements (primal assembly) or by introducing interface forces satisfying the interface equilibrium (dual assembly).

2.3.1 Primal assembly

We can define as primal unknowns for the interface a set of DOF $\mathbf{u}_g$ that are global and uniquely defined for the entire structure. The DOF of each substructure are then obtained by mapping the global set $\mathbf{u}_g$ to the local DOF of each substructure $\mathbf{u}^{(s)}$. Such a mapping was already introduced in the previous section, (2.24), to map the local interface forces to a global set. The same mapping, but now from the global DOF to the local ones can be used to write

$$\mathbf{u}^{(s)} = \mathbf{L}^{(s)}\mathbf{u}_g \qquad \text{or} \qquad \mathbf{u} = \mathbf{L}\mathbf{u}_g \tag{2.32}$$

If the substructure DOF are obtained from a unique set as described above, they automatically satisfy the compatibility conditions that matching interface DOF must be equal. Hence

$$\mathbf{B}\mathbf{L}\mathbf{u}_g = \mathbf{0} \qquad \forall\, \mathbf{u}_g \tag{2.33}$$

This relation, mathematically means that $\mathbf{L}$ represents the nullspace of the constraint matrix $\mathbf{B}$:

$$\boxed{\mathbf{L} = \text{null}(\mathbf{B})} \tag{2.34}$$

► In the example of the beam in Figure 2.2.a for which $\mathbf{L}$ was found in (2.26), this leads to

$$\begin{bmatrix} \begin{bmatrix} \bar{u}_2^{(1)} \end{bmatrix} \\ \begin{bmatrix} \bar{u}_1^{(2)} \\ \bar{u}_2^{(2)} \\ \bar{u}_3^{(2)} \end{bmatrix} \\ \begin{bmatrix} \bar{u}_1^{(3)} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} \tag{2.35}$$

Clearly it can be seen that the substructure DOF are “drawn" from a global and unique set of DOF and thus automatically satisfy the compatibility constraint on the interface: with the compatibility constraint matrix $\mathbf{B}$ obtained in (2.29), one verifies that

[8] In these lecture notes we will introduce the coupling conditions using basically a two field approach. A more general three field approach can also be considered but will not be discussed here. See for instance [303]

$$\mathbf{Bu} = \mathbf{BLu}_g = \left[\begin{bmatrix}1\\0\end{bmatrix}\ \begin{bmatrix}-1&0&0\\0&0&1\end{bmatrix}\ \begin{bmatrix}0\\-1\end{bmatrix}\right]\begin{bmatrix}[1\ 0\ 0]\\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\\ [0\ 0\ 1]\end{bmatrix}\mathbf{u}_g = \mathbf{0}$$

◀

When choosing a unique set of DOF as in (2.32), the coupled problem (2.31) can be simplified and written as

$$\begin{cases}\mathbf{ZLu}_g = \bar{\mathbf{f}} + \bar{\mathbf{g}}\\ \mathbf{L}^T\bar{\mathbf{g}} = \mathbf{0}\end{cases} \tag{2.36}$$

where the compatibility condition is not present anymore since it is automatically verified. This expression is sometimes denoted as the Neumann-Dirichlet form of the coupled problem.

The interface forces $\mathbf{g}$ can be eliminated from this relation by premultiplying the dynamic equilibrium (first line of (2.36)) by $\mathbf{L}^T$, which builds the sum of the equilbrium equations on matching nodes. Considering the equilibrium of the internal forces (second line of (2.36)), the coupled problem can finally be expressed as

$$\boxed{\mathbf{Z}_g\mathbf{u}_g = \mathbf{f}_g \qquad \text{with} \quad \mathbf{Z}_g = \mathbf{L}^T\mathbf{ZL} \quad \text{and} \quad \mathbf{f}_g = \mathbf{L}^T\bar{\mathbf{f}}} \tag{2.37}$$

This form is generally referred to as the *primal assembly* of the coupled problem and matrix $\mathbf{L}^T\mathbf{ZL}$ is the primal assembled impedance of the global system. This assembly is similar to the assembly of finite elements.

► To illustrate this process, consider again the first example in Figure 2.2.a, with the matrix $\mathbf{L}$ given in (2.26). The assembled impedance matrix for this example then writes

$$\begin{aligned}\mathbf{Z}_g = \mathbf{L}^T\mathbf{ZL} &= \left[\begin{bmatrix}1\\0\\0\end{bmatrix}\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\ \begin{bmatrix}0\\0\\1\end{bmatrix}\right]\begin{bmatrix}\begin{bmatrix}Z_{22}^{(1)}\end{bmatrix} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \begin{bmatrix}Z_{11}^{(2)}&Z_{12}^{(2)}&Z_{13}^{(2)}\\Z_{21}^{(2)}&Z_{22}^{(2)}&Z_{23}^{(2)}\\Z_{31}^{(2)}&Z_{32}^{(2)}&Z_{33}^{(2)}\end{bmatrix} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \begin{bmatrix}Z_{11}^{(3)}\end{bmatrix}\end{bmatrix}\begin{bmatrix}[1\ 0\ 0]\\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\\ [0\ 0\ 1]\end{bmatrix}\\ &= \begin{bmatrix}Z_{22}^{(1)}+Z_{11}^{(2)} & Z_{12}^{(2)} & Z_{13}^{(2)}\\ Z_{21}^{(2)} & Z_{22}^{(2)} & Z_{23}^{(2)}\\ Z_{31}^{(2)} & Z_{32}^{(2)} & Z_{11}^{(2)}+Z_{33}^{(3)}\end{bmatrix}\end{aligned} \tag{2.38}$$

and the dynamics of the beam are expressed by

$$\mathbf{Z}_g\mathbf{u}_g = = \mathbf{f}_g \qquad \Longrightarrow \qquad \begin{bmatrix}Z_{22}^{(1)}+Z_{11}^{(2)} & Z_{12}^{(2)} & Z_{13}^{(2)}\\ Z_{21}^{(2)} & Z_{22}^{(2)} & Z_{23}^{(2)}\\ Z_{31}^{(2)} & Z_{32}^{(2)} & Z_{11}^{(2)}+Z_{33}^{(3)}\end{bmatrix}\begin{bmatrix}\bar{u}_1\\ \bar{u}_2\\ \bar{u}_3\end{bmatrix} = \begin{bmatrix}\bar{f}_2^{(1)}+\bar{f}_1^{(2)}\\ \bar{f}_2^{(2)}\\ \bar{f}_3^{(2)}+\bar{f}_1^{(3)}\end{bmatrix} \tag{2.39}$$

◀

2.3.2 Dual assembly

Let us start again from the coupled form (2.30) (or equivalently (2.31)), but instead of satisfying *a priori* the interface compatibility as in the primal assembly, we satisfy *a priori* the interface equilibrium.[9] This can be achieved by choosing coupling forces as

$$\mathbf{g} = -\mathbf{B}^T \boldsymbol{\lambda} \tag{2.40}$$

Note that, since $\mathbf{BL} = \mathbf{0}$ and thus $\mathbf{L}^T\mathbf{B}^T = \mathbf{0}$, such coupling forces automatically satisfy the interface equilibrium:

$$\mathbf{L}^T\mathbf{B}^T\boldsymbol{\lambda} = \mathbf{0} \qquad \forall \boldsymbol{\lambda} \tag{2.41}$$

The minus in this definition is a choice to obtain a symmetric form later. The expression (2.40) can also be written substructure-wise as

$$\begin{bmatrix} \mathbf{g}^{(1)} \\ \vdots \\ \mathbf{g}^{(N^{sub})} \end{bmatrix} = - \begin{bmatrix} \mathbf{B}^{(1)} \\ \vdots \\ \mathbf{B}^{(N^{sub})} \end{bmatrix} \boldsymbol{\lambda} \tag{2.42}$$

showing clearly that the coupling forces are generated from a unique set of interface forces, whose intensities are given by $\boldsymbol{\lambda}$ and are attributed, with the appropriate positive or negative sign, to the substructure interface. The interface force intensities $\boldsymbol{\lambda}$ are unknowns pertaining to the interface. The size n_λ of $\boldsymbol{\lambda}$ is equal to the number of interface compatibility conditions on the interface.

Substituting the coupling forces by their form (2.40), the coupled problem can be written as

$$\boxed{\begin{cases} \mathbf{Z}^{(s)}\bar{\mathbf{u}}^{(s)} = \bar{\mathbf{f}}^{(s)} + \mathbf{B}^{(s)^T}\boldsymbol{\lambda} \qquad s = 1 \dots N^{sub} \\ \displaystyle\sum_{s=1}^{N^{sub}} \mathbf{B}^{(s)}\bar{\mathbf{u}}^{(s)} = \mathbf{0} \end{cases}} \tag{2.43}$$

where the interface equilibrium no longer appears since it is automatically satisfied given our choice (2.40). In block matrix form, we can write

$$\begin{cases} \mathbf{Z}\bar{\mathbf{u}} + \mathbf{B}^T\boldsymbol{\lambda} = \bar{\mathbf{f}} \\ \mathbf{B}\bar{\mathbf{u}} = \mathbf{0} \end{cases} \tag{2.44}$$

which is often put in the symmetric form

$$\boxed{\begin{bmatrix} \mathbf{Z} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix}}. \tag{2.45}$$

One observes that $\boldsymbol{\lambda}$ can be interpreted as the Lagrange multipliers associated with the compatibility constraint (see the general theory of constrained systems in dynamics, for instance in [82]).

► To illustrate this form, consider again the first example in Figure 2.2.a, with the matrix $\mathbf{B}$ given in (2.29). The dually assembled form is then written as

[9] For the description of the problem, it does obviously not manner if one first satisfies the equilibrium or the compatibility. This only determines which equation in the three-field formulation (2.30) is eliminated. The end result will be identical, but the mathematical form and the approximations that can be applied will be depending on the form that will be solved.

$$\begin{bmatrix} \begin{bmatrix} Z_{22}^{(1)} \end{bmatrix} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \mathbf{0} & \begin{bmatrix} Z_{11}^{(2)} & Z_{12}^{(2)} & Z_{13}^{(2)} \\ Z_{21}^{(2)} & Z_{22}^{(2)} & Z_{23}^{(2)} \\ Z_{31}^{(2)} & Z_{32}^{(2)} & Z_{33}^{(2)} \end{bmatrix} & \mathbf{0} & \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix} Z_{11}^{(3)} \end{bmatrix} & \begin{bmatrix} 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \bar{u}_2^{(1)} \end{bmatrix} \\ \begin{bmatrix} \bar{u}_1^{(2)} \\ \bar{u}_2^{(2)} \\ \bar{u}_3^{(2)} \end{bmatrix} \\ \begin{bmatrix} \bar{u}_1^{(3)} \end{bmatrix} \\ \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \bar{f}_2^{(1)} \end{bmatrix} \\ \begin{bmatrix} \bar{f}_1^{(2)} \\ \bar{f}_2^{(2)} \\ \bar{f}_3^{(2)} \end{bmatrix} \\ \begin{bmatrix} \bar{f}_1^{(3)} \end{bmatrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} \tag{2.46}$$

Notice that this representation of the system is larger than the primally assembled counterpart, but in this representation the interface forces, λ are explicitly available. In future sections we shall see that this can be useful.

◀

2.3.3 *Usefulness of different assembly formulations*

In this section, we have expressed the coupled problem in three different forms that are recalled in their block matrix form below for clarity:

General form
$$\begin{cases} \mathbf{Z}\bar{\mathbf{u}} = \bar{\mathbf{f}} + \bar{\mathbf{g}} \\ \mathbf{B}\bar{\mathbf{u}} = \mathbf{0} \\ \mathbf{L}^T\bar{\mathbf{g}} = \mathbf{0} \end{cases} \tag{2.31}$$

Primal Assembly
$$\mathbf{Z}_g\mathbf{u}_g = \mathbf{f}_g \quad \text{with} \quad \mathbf{Z}_g = \mathbf{L}^T\mathbf{Z}\mathbf{L} \quad \text{and} \quad \mathbf{f}_g = \mathbf{L}^T\bar{\mathbf{f}} \tag{2.37}$$

Dual Assembly
$$\begin{bmatrix} \mathbf{Z} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix} \tag{2.45}$$

To illustrate the primal and dual assembly, Figure 2.4 shows the assembly of components with forces or with displacements, and the corresponding matrices.

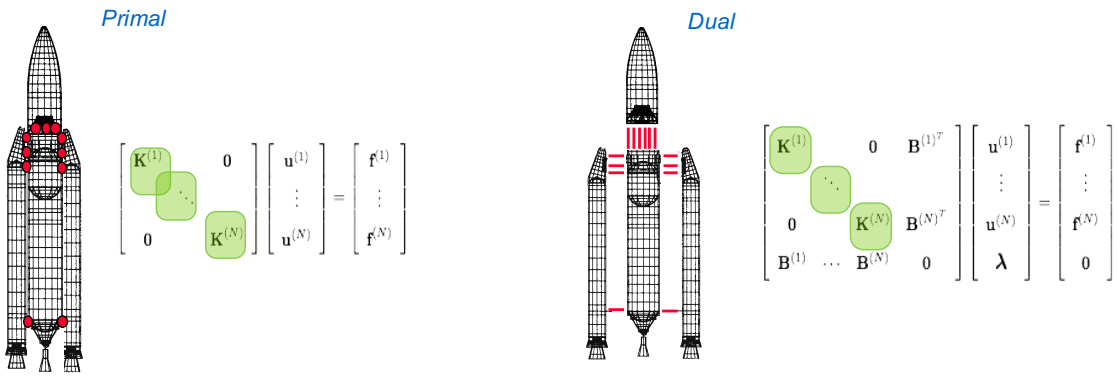


Fig. 2.4: Primal and dual assemble illustration

Each of these representations is mathematically equivalent and, hence, one might wonder why considering one form or the other might be advantageous. From a mechanical point of view the interpretation of these forms is different, and so each provides different insights. The general form considers each substructure as independent (the DOF and coupling forces are defined separately per substructure) and therefore

requires writing all of the interface conditions explicitly. In the primal form, coupling forces are no longer part of the problem: the only interface unknowns are the interface DOF, which are unique for the entire structure. In the dual form, it is the interface force amplitudes $\boldsymbol{\lambda}$ that are unique for the entire structure and represent the interface unknowns (the interface DOF remaining defined per substructure).

The different mechanical interpretations of the assembly can be used to develop different numerical and experimental techniques in substructuring. This will hopefully become clear in the other chapters of these lecture notes, but here we provide a brief preview to the various uses of these forms.

Model Reduction (Chapter 3)

In model reduction by substructuring, the dynamics in a substructure are approximated by representing the behavior of its physical DOF in a subspace of representative modes.

- When the primal assembled form (2.37) is considered, a substructure is seen as receiving information from its neighboring substructures over the interface DOF they share. Hence, a natural set of modes for the reduction might correspond to the static response of the substructure to unit displacements at the interface, and these could be augmented by vibration modes in which the interface is fixed (i.e. the Hurty/Craig-Bampton family of methods naturally arise).
- When the dual assembly form (2.45) is considered, the substructure receives excitation from the neighbors as imposed interface forces. A natural set of representation modes in that case would consist of the static response of the substructure to unit interface forces and the additional dynamics would naturally be represented by vibration modes computed with the interface free.

The reduction methods will thus be based on different approximated representations depending on which mechanical interpretation of the coupling is considered.

Experimental Substructuring (Chapter 4)

In experimental substructuring, the dynamics of (some) substructures are obtained be measurements. Since in experimental dynamics one typically measures a component with free interface, applying excitations and measuring the dynamic response, one typically measures dynamic flexibilities (FRFs), namely the response of the component to excitations. Therefore, considering the dually assembled form seems more natural. Obviously, the measured dynamic flexibility could be inverted and assembled in a primal way (impedance coupling) which would be mathematically equivalent. In practice however, since the measured dynamic flexibility of the substructure always contains imperfections or are not complete, an assembly should usually be performed only in a weak sense : the degrees of freedom on the interface should not be forced to match exactly, but rather only some simple deformation modes on the interface should be made compatible, allowing for some local incompatibility. This alleviates the effect of measurement errors in the assembly as will be explained later (Section 4.3).

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